

1/N phenomenon for some symmetry classes of the odd alternating sign matrices

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Abstract

We consider the alternating sign matrices of the odd order that have some kind of central symmetry. Namely, we deal with matrices invariant under the half-turn, quarter-turn and flips in both diagonals. In all these cases, there are two natural structures in the centre of the matrix. For example, for the matrices invariant under the half-turn the central element is equal ± 1 . It was recently found that $A_{HT}^+(2m+1)/A_{HT}^-(2m+1)=(m+1)/m$. We conjecture that similar very simple relations are valid in the two remaining cases.

An alternating-sign matrix is a matrix with entries 1, 0, and -1 such that the 1 and -1 entries alternate in each column and each row and such that the first and last nonzero entries in each row and column are 1. Starting from the famous conjectures by Mills, Robbins and Rumsey [1, 2] and [3] a lot of enumeration and equinumeration results on alternating-sign matrices and their various subclasses were obtained. Most of the results were proved using bijection between matrices and states of different variants of the statistical square-ice model. For the first time such a method to solve enumeration problems was used by Kuperberg [4], see also the rich in results paper [5].

In the recent paper [6] that was mentioned in the abstract, authors studied enumerations of the half-turn symmetric alternating-sign matrices of odd order on the base of the corresponding square-ice model.

We say that an alternating-sign matrix A of the order n is half-turn symmetric if

$$(A)_{n+1-i, n+1-j} = (A)_{i,j}, \quad i, j = 1, \dots, n.$$

It appears that one can separate the contributions to the partition function of the states corresponding to the alternating-sign matrices having 1 and -1 as the central entry (Theorem 2 of paper [6]). The authors have proved amazingly simple relation

$$A_{HT}^{(+1)}(2m+1)/A_{HT}^{(-1)}(2m+1) = (m+1)/m. \quad (1)$$

In their next paper [7] authors treated the quarter-turn symmetric alternating-sign matrices of odd order and proved the conjectures by Robbins [3] related to enumeration of these matrices.

An alternating-sign $n \times n$ matrix A is said to be quarter-turn symmetric if

$$(A)_{j, n+1-i} = (A)_{i,j}, \quad i, j = 1, \dots, n.$$

We consider the matrices of odd order and write $n = 2m + 1$. Let k^\pm be the numbers of the noncentral matrix elements that are equal ± 1 respectively. It is evident that

$$k^+ + k^- + (A)_{m+1,m+1} = n = 2m + 1.$$

The numbers k^\pm are divisible by 4. One obtains that the quarter-turn symmetric alternating-sign matrix of the order $n = 2m + 1$ has $(A)_{m+1,m+1} = 1$ or $(A)_{m+1,m+1} = -1$ as the central entry according to whether m is even or odd.

In this note we report the additional observation related to the ratio of numbers of these matrices with the different structures in the centre.

In the case $m = 2\mu$, $n = 4\mu + 1$, one has 1 as the central entry. There are two different structures in the centre: four adjacent entries can be 0 or -1 . Using evident notations we can describe our observation as follows:

Conjecture 1a

$$A_{QT}^{(0)}(4\mu + 1)/A_{QT}^{(-1)}(4\mu + 1) = (\mu + 1)/\mu. \quad (2)$$

In the case $m = 2\mu + 1$, $n = 4\mu + 3$, one has -1 as the central entry. Once again, there are two different structures in the centre: four adjacent entries can be 1 or 0. We find now

Conjecture 1b

$$A_{QT}^{(1)}(4\mu + 3)/A_{QT}^{(0)}(4\mu + 3) = (\mu + 1)/\mu. \quad (3)$$

Let us consider the last case - alternating-sign matrices of odd order invariant under flips in both diagonals. Robbins found a pattern for this case [3] but the author does not know the corresponding proof. As in the case of the half-turn symmetric alternating-sign matrices, we have 1 or -1 as the central entry and observe amazingly simple relation

Conjecture 2

$$A_{DD}^{(+1)}(2m + 1)/A_{DD}^{(-1)}(2m + 1) = (m + 1)/m. \quad (4)$$

(Compare with equation (1).)

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